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THE TORSION-EXTENSION COUPLING IN PRETWISTED ELASTIC BEAMS

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Abstract—An explicit asymptotic formula is derived for the untwist of a pretwisted elastic beam subjected to homogeneous extension or equivalently for the longitudinal contraction produced by a torsional moment. It is based on an asymptotic expansion of the three dimensional equations of linear elasticity. The general result consists of two terms, one due to lengthwise variation of the warping of the cross sections and another due to local rotations from bending, when the elastic center is not on the axis of pretwist. The result supports some recent approximate theories for pretwisted elastic beams.

INTRODUCTION

Beams, in which the variation of the cross section is dominated by a continuous rotation with respect to a longitudinal axis, find wide application as rotor blades. In such beams an extensional force will cause untwist, and similarly a torsional moment will cause elongation or contraction depending on whether the original twist is decreased or increased.

In this paper an explicit asymptotic formula is derived for the torsion-extension coupling for an arbitrary homogeneous cross section within the framework of infinitesimal strains. The result is obtained from an asymptotic expansion of the full three dimensional equations of elasticity with the rate of pretwist as the perturbation parameter. It does therefore not contain any a priori assumptions regarding the deformations in the cross sectional plane or the form of the warping function.

The key to the result is the observation that the first order problems are governed by well known field equations of three dimensional elasticity with volume forces and surface loads provided by the zero order solution, i.e. the solution for a similar beam without pretwist. The linear term of the coupling energy can therefore be evaluated explicitly without solving any new boundary value problem for the pretwisted beam. The field equations in local coordinates have formed the basis of special problems treated by Okubo[1, 2] and Goodier and Griffin[3] but do not appear to have been used in the present context before.

Apart from the direct interest of the asymptotic result it may also serve as a means of evaluating the consistency of technical beam theories, where simplifying assumptions have been introduced in order to allow consideration of non-homogeneous loading. Ideally a technical theory for pretwisted elastic beams should be able to reproduce the present coupling term within linear approximation in pretwist and linear definition of strain. This aspect of recent technical beam theories of Rosen[4], Hodges[5] and Krenk[6] is discussed at the end of the paper.

THE FIELD EQUATIONS

Introduce a Cartesian coordinate system $\{x_j'\}$, $j = 1, 2, 3$ and consider an isotropic elastic body bounded by the surface

$$F(x_\alpha) = 0, \quad \alpha = 1, 2 \tag{1}$$

where the coordinates x_α are related to the Cartesian coordinates x_j' by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\beta x_3') & \sin(\beta x_3') \\ -\sin(\beta x_3') & \cos(\beta x_3') \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}. \tag{2}$$

The elastic body described by (1) and (2) is a beam of infinite length and constant cross sectional shape with the rate of pretwist β.

When the beam is in a state of pure torsion or extension the stress state only depends on the local coordinates x_α, $\alpha = 1, 2$. The field equations and boundary conditions are therefore expressed in terms of the local coordinates x_α and the longitudinal coordinate z.

The transformation is straightforward, when the field equations are expressed in index notation and only the indices 1 and 2 are considered as tensor indices. The field equations consist of the equilibrium equations,

$$\sigma_{\alpha'\beta',\beta'} + \sigma_{\alpha'3,3} = 0 \tag{3}$$

$$\sigma_{3\beta',\beta'} + \sigma_{33,3} = 0 \tag{4}$$

and the compatibility equations,

$$(\partial_{\gamma'}\partial_{\gamma'} + \partial_3\partial_3)\sigma_{\alpha'\beta'} + \frac{1}{1+\nu}\Theta_{,\alpha'\beta'} = 0 \tag{5}$$

$$(\partial_{\gamma'}\partial_{\gamma'} + \partial_3\partial_3)\sigma_{3\alpha'} + \frac{1}{1+\nu}\Theta_{,3\alpha'} = 0 \tag{6}$$

$$(\partial_{\gamma'}\partial_{\gamma'} + \partial_3\partial_3)\sigma_{33} + \frac{1}{1+\nu}\Theta_{,33} = 0 \tag{7}$$

where

$$\Theta = \sigma_{\gamma'\gamma'} + \sigma_{33} \tag{8}$$

and ν is Poisson's ratio. A comma denotes differentiation, and summation over the values 1, 2 is implied by repeated Greek indices.

Repeated indices may be replaced directly by their local counterparts on account of (2) leaving only the task of replacing the partial derivative ∂_3 with its local counterpart and transforming the free indices by use of (2). When expressed in local coordinates $\partial_3 f$ will be written as Df, and it follows from (2) that

$$Df = \partial_3 f = (\partial_z - \beta x_\alpha \epsilon_{\alpha\beta}\partial_\beta)f \tag{9}$$

where $\epsilon_{\alpha\beta}$ is the permutation symbol.

In terms of local coordinates the equilibrium equations are

$$\sigma_{\alpha\beta,\beta} + D\sigma_{\alpha 3} - \beta\epsilon_{\alpha\beta}\sigma_{\beta 3} = 0 \tag{10}$$

$$\sigma_{3\beta,\beta} + D\sigma_{33} = 0. \tag{11}$$

The last term in (10) arises from the operator D acting on the transform matrix in the second term. The similar form of the compatibility equations is

$$(\partial_\gamma\partial_\gamma + D^2)\sigma_{\alpha\beta} - 2\beta(\epsilon_{\alpha\gamma}D\sigma_{\gamma\beta} + \epsilon_{\beta\gamma}D\sigma_{\gamma\alpha}) - 2\beta^2(\delta_{\alpha\gamma}\delta_{\beta\delta} - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta})\sigma_{\gamma\delta} + \frac{1}{1+\nu}\Theta_{,\alpha\beta} = 0 \tag{12}$$

$$(\partial_\gamma\partial_\gamma + D^2 - \beta^2)\sigma_{\alpha 3} - 2\beta\epsilon_{\alpha\beta}D\sigma_{\beta 3} + \frac{1}{1+\nu}\partial_\alpha D\Theta = 0 \tag{13}$$

$$(\partial_\gamma\partial_\gamma + D^2)\sigma_{33} + \frac{1}{1+\nu}D^2\Theta = 0. \tag{14}$$

$\delta_{\alpha\beta}$ is Kronecker's delta. The eqns (10)–(14) specialize to equations (4) and (5) of Okubo[2], when ∂_z is omitted from the operator D. These equations were given in a slightly different form by Goodier and Griffin[3].

In the problem considered here the surface is stress free, i.e.

$$\sigma_{\alpha'\beta'}\partial_{\beta'}F + \sigma_{\alpha'3}\partial_3 F = 0 \tag{15}$$

$$\sigma_{3\beta'}\partial_{\beta'}F + \sigma_{33}\partial_3 F = 0. \tag{16}$$

In local coordinates these conditions are

$$\sigma_{\alpha\beta}n_\beta - \beta\sigma_{\alpha 3}x_\beta\epsilon_{\beta\gamma}n_\gamma = 0 \tag{17}$$

$$\sigma_{3\beta}n_\beta - \beta\sigma_{33}x_\beta\epsilon_{\beta\gamma}n_\gamma = 0 \tag{18}$$

where $n_\beta \propto \partial_\beta F$ is the local representation of the normal to the contour in the plane $z = \text{const.}$

TORSION

An asymptotic solution, for small β, to the problem of pure torsion is determined by expanding the local stress components in powers of β.

$$\sigma_{ij}(x_\gamma) = \sum_{k=0}^{\infty} \beta^k \sigma_{ij}^k(x_\gamma), \quad i, j = 1, 2, 3. \tag{19}$$

The zero order solution is the ordinary St. Venant torsion problem, i.e. $\sigma^0_{\alpha\beta} \equiv \sigma^0_{33} \equiv 0$ and

$$\sigma^0_{3\beta,\beta} = 0 \tag{20}$$

$$\sigma^0_{3\alpha,\gamma\gamma} = 0 \tag{21}$$

$$\sigma^0_{3\beta}n_\beta = 0. \tag{22}$$

It is well known—see e.g. [7]—that the stress components $\sigma^0_{3\alpha}$ can be expressed in terms of the warping $w^0(x_\gamma)$ of the cross section in the form

$$\sigma^0_{3\alpha} = G(w^0_{,\alpha} - \epsilon_{\alpha\beta}x_\beta\theta) \tag{23}$$

where θ is the rate of twist caused by the torsional moment, and G is the shear modulus.

In the first order problem the components $\sigma^1_{3\alpha}$ vanish identically, while the remaining stress components satisfy

$$\sigma^1_{\alpha\beta,\beta} + q^1_\alpha = 0 \tag{24}$$

$$\sigma^1_{\alpha\beta,\gamma\gamma} + \frac{1}{1+\nu}\Theta^1_{,\alpha\beta} = 0 \tag{25}$$

$$\sigma^1_{33,\gamma\gamma} = 0 \tag{26}$$

$$\sigma^1_{\alpha\beta}n_\beta = \tau^1_\alpha \tag{27}$$

i.e. the equations of a plane problem of elasticity with the volume force q^1_α and the surface stress τ^1_α determined from the zero order solution.

$$\begin{aligned} q^1_\alpha &= -x_\beta\epsilon_{\beta\gamma}\partial_\gamma\sigma^0_{\alpha 3} - \epsilon_{\alpha\beta}\sigma^0_{\beta 3} \\ &= -G(x_\beta\epsilon_{\beta\gamma}w^0_{,\gamma})_{,\alpha} \end{aligned} \tag{28}$$

$$\tau^1_\alpha = \sigma^0_{\alpha 3}x_\beta\epsilon_{\beta\gamma}n_\gamma. \tag{29}$$

The homogeneous compatibility equations (25) and (26) imply that the stresses σ^1_{ij} can be derived from a displacement field u^1_j by Hooke's law.

EXTENSION

An extensional force must act through the axis of pretwist in order to produce a stress state of the form $\sigma_{ij} = \sigma_{ij}(x_\gamma)$. When the stress field is expanded in powers of β, (19), the zero order solution will therefore in general contain a bending contribution. Let A be the area of the cross section and define its elastic center

$$c_\alpha = \frac{1}{A}\int_A x_\alpha \, dA \tag{30}$$

and the moments of inertia with respect to axes through c_α

$$I_{\alpha\beta} = \int_A (x_\alpha - c_\alpha)(x_\beta - c_\beta)\, dA. \tag{31}$$

When $J_{\alpha\beta}$ designates the inverse of $I_{\alpha\beta}$, the zero order stress field corresponding to an extensional force N is $\sigma^0_{\alpha\beta} \equiv \sigma^0_{\alpha 3} \equiv 0$,

$$\sigma^0_{33} = N\left(\frac{1}{A} - (x_\alpha - c_\alpha)J_{\alpha\beta}c_\beta\right). \tag{32}$$

This stress field can be derived from a displacement field (u^0_α, w^0) by Hooke's law, and it is easily verified that the transverse contraction u^0_α is given by

$$u_\alpha{}^0 = -\nu\frac{N}{E}\left(\left(\frac{1}{A} + c_\beta J_{\beta\gamma}c_\gamma\right)x_\alpha - \frac{1}{2}x_\alpha x_\beta J_{\beta\gamma}c_\gamma - \frac{1}{2}\epsilon_{\alpha\gamma}x_\gamma x_\delta \epsilon_{\delta\phi}J_{\phi\beta}c_\beta\right) \tag{33}$$

where E is the modulus of elasticity. In particular the rotation is

$$u^0_{\alpha,\beta}\epsilon_{\alpha\beta} = 2\,\nu\frac{N}{E}x_\alpha \epsilon_{\alpha\gamma}J_{\gamma\beta}c_\beta. \tag{34}$$

In the first order problem the components $\sigma^1_{\alpha\beta}$ and σ^1_{33} vanish identically, while the remaining components satisfy

$$\sigma^1_{3\beta,\beta} + q^1_3 = 0 \tag{35}$$

$$\sigma^1_{3\alpha,\gamma\gamma} = 0 \tag{36}$$

$$\sigma^1_{3\alpha}n_\alpha = \tau^1_3 \tag{37}$$

where the volume force q^1_3 is

$$q^1_3 = -x_\beta \epsilon_{\beta\gamma}\sigma^0_{33,\gamma} \tag{38}$$

and the surface stress τ^1_3 is

$$\tau^1_3 = \sigma^0_{33}x_\beta \epsilon_{\beta\gamma}n_\gamma. \tag{39}$$

Also in this case the homogeneity of the compatibility equation (36) implies that the stresses σ^1_{ij} can be derived from a displacement field u^1_j by Hooke's law.

TORSION-EXTENSION COUPLING

Consider the part of the beam located between the planes $z = 0$ and $z = l$. Application of a torsional moment M conforming with the stress distribution that is independent of z will produce an elongation of magnitude $l e_M$. Similarly application of an extensional force through the axis of pretwist conforming with the stress distribution that is independent of z will produce an additional twist $l\theta_N$.

The first term in an asymptotic expansion of e_M and θ_N follows from the reciprocity relation. Let a tilde designate stresses and displacements produced by the torsional moment M, and let an overbar designate stresses and displacements produced by the extensional force N. The reciprocity relation then gives

$$lNe_M = lM\theta_N = l\int_A \tilde{\sigma}_{ij}\bar{u}_{i,j}\,\mathrm{d}A$$
$$= l\int_A \{\tilde{\sigma}^0_{ij}\bar{u}^0_{i,j} + \beta(\tilde{\sigma}^1_{ij}\bar{u}^0_{i,j} + \tilde{\sigma}^0_{ij}\bar{u}^1_{i,j}) + O(\beta^2)\}\,\mathrm{d}A. \quad (40)$$

The first term vanishes, leaving

$$Ne_M = M\theta_N = \beta\int_A (\bar{\sigma}^1_{ij}\tilde{u}^0_{i,j} + \tilde{\sigma}^1_{ij}\bar{u}^0_{i,j})\,\mathrm{d}A$$
$$= \beta\left\{\int_A (\bar{q}^1_3\tilde{w}^0 + \tilde{q}^1_\alpha\bar{u}^0_\alpha)\,\mathrm{d}A + \int_C (\bar{\tau}^1_3\tilde{w}^0 + \tilde{\tau}^1_\alpha\bar{u}^0_\alpha)\,\mathrm{d}S\right\} \quad (41)$$

where the divergence theorem has been used in connection with (24), (27), (35) and (37).

Substitution of volume forces and surface stresses from (28), (29), (38) and (39) gives

$$Ne_M = M\theta_N = \beta\left\{\int_A [-x_\gamma\epsilon_{\gamma\beta}\bar{\sigma}^0_{33,\beta}\tilde{w}^0 - G(x_\gamma\epsilon_{\gamma\beta}\tilde{w}^0_{,\beta})_{,\alpha}\bar{u}^0_\alpha]\,\mathrm{d}A\right.$$
$$\left. + \int_C (\bar{\sigma}^0_{33}\tilde{w}^0 + \tilde{\sigma}^0_{\alpha 3}\bar{u}^0_\alpha)x_\beta\epsilon_{\beta\gamma}n_\gamma\,\mathrm{d}s. \right. \quad (42)$$

The last term in the contour integral can be reformulated by using that on the contour $\tilde{\sigma}^0_{\alpha 3}n_\alpha \equiv 0$. This implies the following identity on the contour.

$$\tilde{\sigma}^0_{\alpha 3}\bar{u}^0_\alpha x_\beta\epsilon_{\beta\gamma}n_\gamma = x_\alpha\tilde{\sigma}^0_{\alpha 3}\bar{u}^0_\beta\epsilon_{\beta\gamma}n_\gamma. \quad (43)$$

When this is introduced in (42) and $\tilde{\sigma}^0_{\alpha 3}$ is eliminated by (23), the divergence theorem reduces (42) to an area integral.

$$Ne_M = M\theta_N = \beta\int_A \{(x_\alpha\epsilon_{\alpha\beta}\tilde{w}^0_{,\beta})\bar{\sigma}^0_{33} + Gx_\alpha\tilde{w}^0_{,\alpha}\bar{u}^0_{\beta,\gamma}\epsilon_{\beta\gamma}$$
$$+ Gx_\alpha(\tilde{w}^0_{,\alpha\gamma}\epsilon_{\beta\gamma} - \tilde{w}^0_{,\gamma\beta}\epsilon_{\alpha\gamma})\bar{u}^0_\beta\}\,\mathrm{d}A. \quad (44)$$

The last term in (44) vanishes, because $\tilde{w}^0_{,\gamma\gamma} = 0$ in order to satisfy (20). Thus the final result is

$$Ne_M = M\theta_N = \beta\int_A \{(x_\alpha\epsilon_{\alpha\beta}\tilde{w}^0_{,\beta})\bar{\sigma}^0_{33} + Gx_\alpha\tilde{w}^0_{,\alpha}\bar{u}^0_{\beta,\gamma}\epsilon_{\beta\gamma}\}\,\mathrm{d}A. \quad (45)$$

The first term in (45) is simply the longitudinal strain introduced by the rotation of the warped cross sections, while the last term is the local rotation introduced by bending. The factors $\bar{\sigma}^0_{33}$ and $\bar{u}^0_{\beta,\gamma}$ can be introduced explicitly from (32) and (34), yielding

$$Ne_M = M\theta_N = \beta N\int_A \left\{x_\alpha\epsilon_{\alpha\beta}\tilde{w}^0_{,\beta}\left(\frac{1}{A} - (x_\gamma - c_\gamma)J_{\gamma\delta}c_\delta\right) + \frac{\nu}{1+\nu}x_\alpha\tilde{w}^0_{,\alpha}(x_\beta\epsilon_{\beta\gamma}J_{\gamma\delta}c_\delta)\right\}\mathrm{d}A. \quad (46)$$

In the special case where the elastic center is on the axis of pretwist, i.e. $c_\alpha = 0$, the result simplifies considerably and can be expressed in three different ways. The first is simply the first term of (46). The second is obtained by expressing $\tilde{w}^0_{,\beta}$ in terms of $\tilde{\sigma}^0_{3\alpha}$ by (23) and introducing the torsional constant K by $M = GK\theta$, while the third follows from the orthogonality of $\tilde{\sigma}^0_{3\alpha}$ and $\tilde{w}^0_{,\alpha}$. For $c_\alpha = 0$ (46) can then be written as

$$Ne_M = M\theta_N = \beta \frac{N}{A}\int_A x_\alpha \epsilon_{\alpha\beta}\tilde{w}^0_{,\beta}\, dA$$
$$= -\frac{\beta}{G}\frac{N}{A}\frac{M}{K}(I_{\alpha\alpha} - K)$$
$$= -\beta G\frac{N}{A}\frac{K}{M}\int_A \tilde{w}^0_{,\alpha}\tilde{w}^0_{,\alpha}\, dA. \qquad (47)$$

The last integral shows that for $c_\alpha = 0$ the coupling is negative except for sections without warping, i.e. circular sections, where there is no coupling. The second expression conveniently gives the coupling in terms of the polar moment of inertia, $I_{\alpha\alpha}$, and the torsional constant, K. The first expression is an integral of the longitudinal strain introduced into the torsion problem by the rotation of the warping function and is the form found in the general formula (45).

DISCUSSION

The assumption of a stress state described completely in terms of the local coordinates x_α is too restrictive for many practical problems involving, e.g. nonhomogeneous loading, bending and variable cross section. As analytical solutions, even in asymptotic form, beyond homogeneous torsion and extension[1, 2] and homogeneous bending[3] seems to be unobtainable, it is important to develop technical theories to deal with the general problem in an approximate way. A common way of doing this consists in assuming a suitably simple displacement field and then applying the principle of minimum potential energy or the principle of virtual work with an assumed or derived stress field.

In [4] Rosen considered the problem of homogeneous torsion of a pretwisted beam by the method of potential energy. The assumed displacement field included the St. Venant warping function, i.e. $\tilde{w}^0$, and although the extensional coupling was not discussed explicitly in [4], it appears in the differential equations and can be identified with the first form of (47). Thus, if either the pretwist is around the elastic axis, or if $\nu = 0$, the coupling term contained in [4] is asymptotically correct. The present result is limited to linear terms in β and therefore does not give direct information about the torsional stiffness as a function of pretwist. However, it does follow that, if transverse deformations of the cross sections are neglected, as is often the case, and if only one function is used to describe warping of the cross section, then this function should be the St. Venant warping function $\tilde{w}^0$ in order to reproduce the linear coupling term.

The analysis by Hodges[5] of the torsion-extension coupling of pretwisted elastic beams was based on the principle of virtual work. In addition of an assumed displacement field it therefore also requires a stress field. In providing this it was assumed that the transverse stress components with respect to the beam axis could be neglected. This assumption is equivalent to that contained in the potential energy approach of Rosen[4]. However, two extensions were made in [5]: the magnitude of the warping was not related to the twist a priori, and finite strains were used. The torsion-extension coupling obtained by Hodges, who initially assumed $c_\alpha = 0$, contains two terms, of which one is linear and the other quadratic in the strains. The linear term corresponds to the second form of (47) and thus is asymptotically correct.

Recently a technical theory for pretwisted elastic beams of arbitrary cross section under general loads was developed in [6] by use of an assumed displacement field including the St. Venant warping function. The theory makes a clear distinction between the elastic axis, the axis through the shear center and the axis of pretwist. Only the latter is assumed to be straight. This theory reproduces the first half of the general result (46) and thereby gives the asymptotically correct torsion-extension coupling apart from the term due to local rotations caused by bending. Naturally this term can not be reproduced by any theory that neglects the deformations of the cross sections in their own plane.

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